

# Magnetic resonance spectroscopy and characterization of magnetic phases for spinor Bose-Einstein condensates

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The response of spinor Bose-Einstein condensates to dynamical modulation of magnetic fields is discussed with the linear response theory. As an experimentally measurable quantity, the energy absorption rate (EAR) is considered, and the response function is found to access quadratic spin correlations which come from the perturbation of the quadratic Zeeman term. By applying our formalism to spin-1 condensates, we demonstrate that the EAR spectrum as a function of the modulation frequency is able to characterize the different magnetically ordered phases.

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*Introduction.*— Ultracold Bose atoms with spin degrees of freedom [1–3] have been attracting interests as a class of quantum fluids accompanying non-trivial spin orders and topological spin textures, in contrast with spinless Bose-Einstein condensates (BECs). In BECs with spins, so-called spinor BECs, spin rotational symmetry allows spin dependent interactions, and the number of the independent interactions increases with spin degrees of freedom of atoms. Furthermore, such spin dependent interactions cause various ground states. However, in addition to exploring the properties of those nontrivial spin orders, it is also important to specify them experimentally. Thus the development of the measurement techniques to capture complicated orders is a challenge for the study on spinor BECs.

There exist the several experimental techniques for spinor BECs: the spin texture imaging by the separation of the different spin components by magnetic field gradient, the spatially-resolved magnetometer by observing Larmor precession [4–6], and measurement of phase transitions and spin dynamics by dynamical control of quadratic Zeeman (QZ) shift [7–10]. In addition, measurement of birefringence by the off-resonant light scattering technique has been analysed, which can access structure of spin textures such as nematicity and Skyrmions. [5, 11] These probes are mainly used to capture static properties of systems, and it is not easy to capture excitation spectra involving spin degrees of freedom.

For spinor BECs in the presence of uniform magnetic fields, the QZ shift in addition to the linear Zeeman (LZ) shift emerges due to hyperfine couplings between a nuclear and an electron spin. [12] The LZ shift in experiments is considered to be negligible because of spin conservation, and indeed the magnetization of spinor BECs is preserved at least within the limit of accuracy of experimental errors. [13] Thus the most relevant effect induced by the magnetic field is the QZ shift, and furthermore what is important is that this QZ coupling is an experimentally controllable parameter. [6, 8, 9, 14].

In this paper, motivated by such possible control of the QZ coupling, we consider magnetic resonant spectra

as a response to dynamically modulated magnetic fields. As an experimentally measurable quantity, we focus on energy absorption rate (EAR), and formulate this EAR spectrum as a function of the modulation frequency of magnetic fields by using linear response theory. In addition, to demonstrate the potential application of this spectroscopy, we analyse the spectra for uniform spin-1 BECs with Bogoliubov theory [15, 16], and the consequent spectra are found to exhibit different behaviors in each phase. Finally we also consider the cases in the presence of trap potential, and conclude that the ordered phases of the trapped systems remains distinguishable from the low-frequency behavior of the EAR spectrum.

*Formalism.*— We start with general spin- $F$  Bose atom systems under an uniform magnetic field. Let us suppose the many-body static Hamiltonian including Zeeman couplings to be  $H_0$ . In this paper, we restrict ourselves that the magnetic field is applied along  $z$  axis, and the Hamiltonian is invariant under spin rotation around  $z$  axis, which is a general setup of the experiments.

In the presence of the dynamically modulated magnetic field such as  $h + \delta h \cos(\omega t/2)$ , the system should be described by the time-dependent Hamiltonian  $H(t) = H_0 + V(t)$ , and the perturbation is represented as

$$V(t) = (\delta p) \cos(\omega t/2) \mathcal{F}_L + (\delta q) \cos^2(\omega t/2) \mathcal{F}_Q, \quad (1)$$

where the first and second terms mean modulation of the LZ and QZ couplings, respectively. The coupling constants,  $\delta p$  and  $\delta q$ , are proportional to  $\delta h$  and  $(\delta h)^2$ , respectively. The LZ and QZ operators are represented as

$$\mathcal{F}_L = \int d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) F^z \hat{\Psi}(\mathbf{r}), \quad (2)$$

$$\mathcal{F}_Q = \int d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) (F^z)^2 \hat{\Psi}(\mathbf{r}), \quad (3)$$

where  $\hat{\Psi} = (\psi_F, \psi_{F-1}, \dots, \psi_{-F})^T$  denotes a spinor boson field, and  $\mathbf{F} = (F^x, F^y, F^z)$  is a spin- $F$  matrix.

In experiments, the EAR can be measured through the time of flight image. Assuming the energy scale of the periodically modulated perturbation to be small

enough [17], the dynamics is well-described with the linear response theory. Then, the EAR is defined as  $R(\omega) = \frac{1}{2\pi/\omega} \int_T^{T+2\pi/\omega} dt \frac{d\langle H(t) \rangle}{dt}$  where  $\langle \dots \rangle$  denotes the statistical average by  $H(t)$ . Thus the EAR is derived as

$$R(\omega) = -\frac{1}{2\hbar} \omega \text{Im}[\tilde{\chi}^R(\omega)], \quad (4)$$

where  $\tilde{\chi}^R(\omega)$  is the Fourier transformed retarded correlation function of the perturbation (1):  $\chi^R(t) = -i \langle [V(t), V(0)] \rangle_0$  averaged by  $H_0$ . Since  $H_0$  is assumed to possess the spin rotational symmetry around  $z$  axis,  $[\mathcal{F}_L, H_0] = [\mathcal{F}_L, \mathcal{F}_Q] = 0$ , and thus the retarded correlation function is reduced to

$$\tilde{\chi}^R(\omega) = -i(\delta q)^2 \int_0^\infty dt e^{i\omega t} \langle [\mathcal{F}_Q(t), \mathcal{F}_Q(0)] \rangle_0. \quad (5)$$

Namely, the system is insensitive to the dynamic modulation of the LZ coupling.

*EAR for spin-1 BECs.*— Let us demonstrate the EAR spectrum (4) to allow for characterizing spin-ordered phases. We consider spin-1 interacting bosons without a trap, which undergo BEC in the low-temperature regime. The Hamiltonian to be considered is given by

$$H_0 = \int d\mathbf{r} \left[ -\frac{\hbar^2}{2M} \hat{\Psi}^\dagger(\mathbf{r}) \nabla^2 \hat{\Psi}(\mathbf{r}) + q \hat{\Psi}^\dagger(\mathbf{r}) (F^z)^2 \hat{\Psi}(\mathbf{r}) + \frac{c_0}{2} \left( \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \right)^2 + \frac{c_1}{2} \left( \hat{\Psi}^\dagger(\mathbf{r}) \mathbf{F} \hat{\Psi}(\mathbf{r}) \right)^2 \right], \quad (6)$$

where  $\hat{\Psi} = (\psi_1, \psi_0, \psi_{-1})^T$ ,  $M$  denotes a mass of atoms, and  $c_0$  and  $c_1$  mean the density and the spin exchange interaction, respectively. For  $^{23}\text{Na}$  and  $^{87}\text{Rb}$  atoms, the coupling  $c_1$  is taken to be positive and negative value, respectively. Let us impose  $nc_0 \gg n|c_1|, |q|$ , where  $n$  is a atom density to meet the experimental conditions. Since the EAR is independent of the LZ coupling modulation as discussed above, instead of (1) we here suppose the modulation perturbation of the magnetic field to be

$$V(t) = (\delta q) \cos(\omega t) \mathcal{F}_Q. \quad (7)$$

The mean-field (MF) analysis, in which the field  $\hat{\Psi}$  is replaced by a MF spinor order parameter  $\hat{\xi}$  optimizing the Hamiltonian (6), leads to the following ground states [12] as shown in Fig. 1: (i) the ferromagnetic (FM) phase  $\hat{\xi}_{\text{FM}} = (1, 0, 0)^T$  for  $c_1, q < 0$ , (ii) the longitudinal polar (LP) phase  $\hat{\xi}_{\text{LP}} = (0, 1, 0)^T$  for  $c_1, q > 0$  or for  $c_1 < 0$  and  $q > 2n|c_1|$ , (iii) the transverse polar (TP) phase  $\hat{\xi}_{\text{TP}} = (1/\sqrt{2}, 0, 1/\sqrt{2})^T$  for  $c_1 > 0$  and  $q < 0$ , and (iv) the BA phase  $\hat{\xi}_{\text{BA}} = (\sin \theta/\sqrt{2}, \cos \theta, \sin \theta/\sqrt{2})^T$ , where  $\sin \theta = \sqrt{(1 - \tilde{q})/2}$  with  $\tilde{q} = q/(2n|c_1|)$ , for  $c_1 < 0$  and  $0 < q < 2n|c_1|$ .

The correlation function  $\tilde{\chi}^R(\omega)$  is calculated with the Bogoliubov theory, by the replacement  $\hat{\Psi} = \sqrt{N_0} \hat{\xi}_\alpha + \delta \hat{\Psi}$  ( $\alpha = \text{FM, LP, TP, BA}$ ) where  $N_0$  is the condensate atom

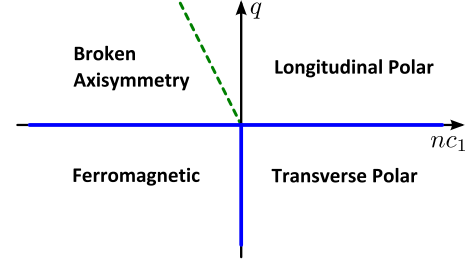


FIG. 1. (Color online) The MF phase diagram of spin-1 BECs. The phase boundary between the BA and LP phase is given by  $q = 2n|c_1|$ . The thick-solid and dashed boundaries denote a first and second order phase transition, respectively.

number. Then the Bogoliubov Hamiltonian is represented as  $H_0 \approx E_{\text{MF}} + H_{\text{eff}}$  with the MF energy  $E_{\text{MF}}$  and

$$H_{\text{eff}} = \sum_{\mathbf{k} \neq 0} \left[ \hat{a}_{\mathbf{k}}^\dagger \left( \epsilon_{\mathbf{k}} + nc_1 \overline{\mathbf{F}} \cdot (\mathbf{F} - \overline{\mathbf{F}}) + q((F^z)^2 - \overline{(F^z)^2}) \right) \hat{a}_{\mathbf{k}} + \frac{nc_0}{2} (D_{\mathbf{k}}^\dagger D_{\mathbf{k}} + D_{\mathbf{k}} D_{-\mathbf{k}} + \text{H.c.}) + \frac{nc_1}{2} (\mathbf{S}_{\mathbf{k}}^\dagger \cdot \mathbf{S}_{\mathbf{k}} + \mathbf{S}_{\mathbf{k}} \cdot \mathbf{S}_{-\mathbf{k}} + \text{H.c.}) \right], \quad (8)$$

where  $\hat{a}_{\mathbf{k}} = \int d\mathbf{r} e^{i\mathbf{k} \cdot \mathbf{r}} \delta \hat{\Psi} = (a_{1\mathbf{k}}, a_{0\mathbf{k}}, a_{-1\mathbf{k}})^T$  is a Fourier transform of the fluctuation of the spinor.  $D_{\mathbf{k}} = \hat{\xi}^\dagger \hat{a}_{\mathbf{k}}$  and  $\mathbf{S}_{\mathbf{k}} = \hat{\xi}^\dagger \mathbf{F} \hat{a}_{\mathbf{k}}$  mean a density and spin fluctuations, respectively, and  $\overline{X} = \hat{\xi}^\dagger X \hat{\xi}$  for a spin operator  $X$  denotes the MF average of spin matrices. In this representation, the QZ operator is expressed as

$$\mathcal{F}_Q = N \overline{(F^z)^2} + \sum_{\mathbf{k} \neq 0} \hat{a}_{\mathbf{k}}^\dagger \left[ (F^z)^2 - \overline{(F^z)^2} \right] \hat{a}_{\mathbf{k}}. \quad (9)$$

*FM phase.*— The MF spinor leads to the following density and spin fluctuation as  $D_{\mathbf{k}} = S_{\mathbf{k}}^z = a_{1\mathbf{k}}$ ,  $S_{\mathbf{k}}^x = a_{0\mathbf{k}}/\sqrt{2}$ , and  $S_{\mathbf{k}}^y = ia_{0\mathbf{k}}/\sqrt{2}$ . The effective Hamiltonian diagonalized by the Bogoliubov transformation is given [16] as

$$H_{\text{eff}}^{\text{FM}} = \sum_{\mathbf{k} \neq 0} \left[ E_{\text{d}}^{\text{FM}}(\mathbf{k}) d_{\text{F}}^\dagger(\mathbf{k}) d_{\text{F}}(\mathbf{k}) + E_z^{\text{FM}}(\mathbf{k}) f_z^\dagger(\mathbf{k}) f_z(\mathbf{k}) + E_{xy}^{\text{FM}}(\mathbf{k}) f_{xy}^\dagger(\mathbf{k}) f_{xy}(\mathbf{k}) \right], \quad (10)$$

where  $E_{\text{d}}^{\text{FM}}(\mathbf{k}) = \sqrt{\epsilon_{\mathbf{k}}[\epsilon_{\mathbf{k}} + 2n(c_0 - |c_1|)]}$ ,  $E_z^{\text{FM}}(\mathbf{k}) = \epsilon_{\mathbf{k}} + 2|c_1|$ , and  $E_{xy}^{\text{FM}}(\mathbf{k}) = \epsilon_{\mathbf{k}} + |q|$ . According to the Bogoliubov transform, the QZ operator in the FM phase is written as

$$\mathcal{F}_Q^{\text{FM}} = \text{Const.} + \sum_{\mathbf{k} \neq 0} f_{xy}^\dagger(\mathbf{k}) f_{xy}(\mathbf{k}). \quad (11)$$

The perturbation  $\mathcal{F}_Q^{\text{FM}}$  commutes with the Hamiltonian (10), and immediately the response function  $\chi^R(\omega)$

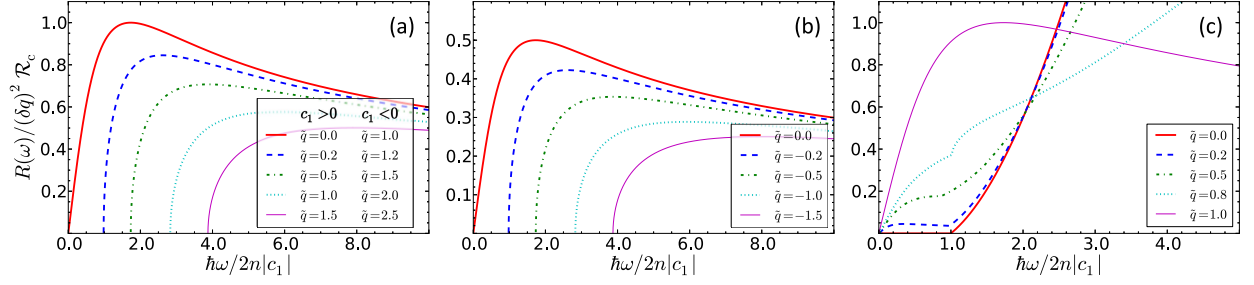


FIG. 2. (Color online) The EAR spectra as a function of modulation frequency for the different QZ couplings  $\tilde{q} = q/2n|c_1|$ : (a) the LP phase for  $c_1, q > 0$  or for  $c_1 < 0$  and  $q > 2n|c_1|$ , (b) the TP phase for  $c_1 > 0$  and  $q < 0$ , and (c) the BA phase for  $c_1 < 0$  and  $0 < q < 2n|c_1|$ . The spectra (a) and (b) show qualitatively the same behavior, but the intensity differs quantitatively due to the different number of the accessible gapful-spin modes. The spectra (c) apart from  $\tilde{q} = 1$  show the abrupt enhancement of the EAR around  $\hbar\omega \approx 2n|c_1|$ , which indicates the appearance of the contribution from  $r_d^{\text{BA}}$ .

is concluded to be zero. Thus the EAR spectrum in the FM phase shows no signal in the entire  $\omega$  regime.

*LP phase.*— The MF spinor  $\hat{\xi}_{\text{LP}}$  perpendicular to the magnetic field leads to  $D_{\mathbf{k}} = a_{0\mathbf{k}}$ ,  $S_{\mathbf{k}}^x = (a_{1\mathbf{k}} + a_{-1\mathbf{k}})/\sqrt{2}$ ,  $S_{\mathbf{k}}^y = i(a_{1\mathbf{k}} - a_{-1\mathbf{k}})/\sqrt{2}$ , and  $S_{\mathbf{k}}^z = 0$ . The diagonalized Bogoliubov Hamiltonian is given [16] as

$$H_{\text{eff}}^{\text{LP}} = \sum_{\mathbf{k} \neq 0} \left[ E_{\mathbf{k}} d^\dagger(\mathbf{k}) d(\mathbf{k}) + \sum_{\nu=x,y} E_{\mathbf{k}}^f f_\nu^\dagger(\mathbf{k}) f_\nu(\mathbf{k}) \right], \quad (12)$$

where  $E_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}} + 2nc_0)}$  and  $E_{\mathbf{k}}^f = \sqrt{(\epsilon_{\mathbf{k}} + q)(\epsilon_{\mathbf{k}} + q + 2nc_1)}$  are, respectively, a gapless-phonon and doubly-degenerate gapful-spin mode. The Bogoliubov transform gives the following form of the QZ perturbation as

$$\mathcal{F}_{\text{Q}}^{\text{LP}} = \bar{\mathcal{F}}_{\text{Q}}^{\text{LP}} - \sum_{\mathbf{k} \neq 0} \sum_{\nu=x,y} \mathcal{A}_{\mathbf{k}}^{\text{LP}} \left[ f_\nu^\dagger(\mathbf{k}) f_\nu^\dagger(-\mathbf{k}) + \text{H.c.} \right], \quad (13)$$

where  $\bar{\mathcal{F}}_{\text{Q}}^{\text{LP}}$  contains the terms commutable with the Hamiltonian (12), and  $\mathcal{A}_{\mathbf{k}}^{\text{LP}} = \sqrt{(\epsilon_{\mathbf{k}} + q + nc_1)^2 - (E_{\mathbf{k}}^f)^2}/2E_{\mathbf{k}}^f$ . The QZ perturbation accesses the two spin modes. From Eq. (13), the retarded correlation function is straightforwardly calculated, and consequently the EAR spectrum per unit volume is analytically obtained as  $R^{\text{LP}}(\omega) = (\delta q)^2 \mathcal{R}_c \theta_{\text{H}}(|\tilde{\omega}| - 2\Delta_{\tilde{q}}) r^{\text{LP}}(\omega)$  with

$$r^{\text{LP}}(\omega) = 2\sqrt{\frac{\sqrt{1 + \tilde{\omega}^2} - 2\tilde{q} - \text{sgn}(c_1)}{1 + \tilde{\omega}^2}}, \quad (14)$$

where  $\theta_{\text{H}}(x)$  is a Heaviside's step function, and  $\mathcal{R}_c = \Omega(2M|c_1|)^{3/2}/64\pi\hbar^4$  with the system volume  $\Omega$  is a constant. Here, the following normalizations have been used:  $\tilde{\omega} = \hbar\omega/2n|c_1|$ ,  $\tilde{q} = q/2n|c_1|$ , and  $\Delta_{\tilde{q}} = \sqrt{\tilde{q}(\tilde{q} + \text{sgn}(c_1))}$  which means a spin gap normalized by  $2nc_1$ . Equation (14) for various values of  $q$  is plotted in Fig. 2(a). We

note that the gap of the EAR spectrum,  $2\Delta_{\tilde{q}}$ , closes on the phase boundaries,  $q = 0$  with  $c_1 > 0$  and  $q = 2n|c_1|$  with  $c_1 < 0$ .

*TP phase.*— The MF spinor  $\hat{\xi}_{\text{TP}}$  provide the fluctuations  $D_{\mathbf{k}} = (a_{1\mathbf{k}} + a_{-1\mathbf{k}})/\sqrt{2}$ ,  $S_{\mathbf{k}}^x = a_{0\mathbf{k}}$ ,  $S_{\mathbf{k}}^y = 0$ , and  $S_{\mathbf{k}}^z = (a_{1\mathbf{k}} - a_{-1\mathbf{k}})/\sqrt{2}$ , and the Bogoliubov Hamiltonian is diagonalized [16] as

$$H_{\text{eff}}^{\text{TP}} = \sum_{\mathbf{k} \neq 0} \left[ E_{\mathbf{k}} d^\dagger(\mathbf{k}) d(\mathbf{k}) + E_{\mathbf{k}}^z f_z^\dagger(\mathbf{k}) f_z(\mathbf{k}) + E_{\mathbf{k}}^f f^\dagger(\mathbf{k}) f(\mathbf{k}) \right], \quad (15)$$

where  $E_{\mathbf{k}}$  and  $E_{\mathbf{k}}^z = \sqrt{\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}} + 2c_1)}$  are gapless modes of phonon and spin, respectively, and  $E_{\mathbf{k}}^f$  is another spin mode with the gap  $\Delta_{\tilde{q}}$ . The QZ operator is written as

$$\mathcal{F}_{\text{Q}}^{\text{TP}} = \bar{\mathcal{F}}_{\text{Q}}^{\text{TP}} - \sum_{\mathbf{k} \neq 0} \mathcal{A}_{\mathbf{k}}^{\text{TP}} \left[ f^\dagger(\mathbf{k}) f^\dagger(-\mathbf{k}) + \text{H.c.} \right], \quad (16)$$

where  $\bar{\mathcal{F}}_{\text{Q}}^{\text{TP}}$  denotes the terms commutable with the Hamiltonian (15), and  $\mathcal{A}_{\mathbf{k}}^{\text{TP}} = \sqrt{(\epsilon_{\mathbf{k}} + |q| + c_1)^2 - (E_{\mathbf{k}}^f)^2}/2E_{\mathbf{k}}^f$ . From Eq. (16), the QZ modulation is found to access only the gapful-spin mode. Straightforwardly the retarded correlation function of Eq. (16) is calculated and the EAR is analytically obtained as  $R^{\text{TP}}(\omega) = (\delta q)^2 \mathcal{R}_c \theta_{\text{H}}(|\tilde{\omega}| - 2\Delta_{\tilde{q}}) r^{\text{TP}}(\omega)$  with

$$r^{\text{TP}}(\omega) = \sqrt{\frac{\sqrt{1 + \tilde{\omega}^2} - 2|\tilde{q}| - 1}{1 + \tilde{\omega}^2}}, \quad (17)$$

which is illustrated in Fig. 2.

It is remarkable that in the positive  $c_1$  regime we have  $R^{\text{LP}}(\omega) = 2R^{\text{TP}}(\omega)$  for the fixed  $|q|$ , and the factor 2 comes from the accessible number of the spin modes by the QZ perturbation; namely the perturbation (13) for the TP phase involves the two gapful-spin modes, while only one gapful-spin mode is accessed for the LP phase. Thus, the qualitative difference of the spectrum intensity allows us to distinguish the two polar phase. In addition,

for this different spectrum intensity we find the following result: if we continuously change  $q$  across  $q = 0$ , the discontinuous spectrum change is observed. This would be interpreted to be a signal of the first order phase transition between the polar phases.

In summary, the EAR in the polar phases has two important features: The first is that we can measure the spin gap  $\Delta_{\tilde{q}}$  in the polar phases. In particular this spin gap is indeed a dominant feature of the low-energy spin excitation. The second is that by the discontinuous difference of the spectrum intensity, we can observe the first order phase transition from the dynamical viewpoint. As we will discuss later, these conclusions do not change even if we have trap potentials.

*BA phase.*— The MF spinor  $\hat{\xi}_{\text{BA}}$  provides  $D_{\mathbf{k}} = \sin\theta(a_{1\mathbf{k}} + a_{-1\mathbf{k}})/\sqrt{2} + \cos\theta a_{0\mathbf{k}}$ ,  $S_{\mathbf{k}}^x = \cos\theta(a_{1\mathbf{k}} + a_{-1\mathbf{k}})/\sqrt{2} + \sin\theta a_{0\mathbf{k}}$ ,  $S_{\mathbf{k}}^y = i\cos\theta(a_{1\mathbf{k}} - a_{-1\mathbf{k}})/\sqrt{2}$ , and  $S_{\mathbf{k}}^z = \sin\theta(a_{1\mathbf{k}} - a_{-1\mathbf{k}})$ , and under the condition  $c_0 \gg |c_1|$  the Bogoliubov Hamiltonian is diagonalized [15, 16, 18] as

$$H_{\text{eff}}^{\text{BA}} = \sum_{\mathbf{k} \neq 0} \left[ E_{\text{d}}^{\text{BA}}(\mathbf{k}) d^\dagger(\mathbf{k}) d(\mathbf{k}) + E_z^{\text{BA}}(\mathbf{k}) f_z^\dagger(\mathbf{k}) f_z(\mathbf{k}) + E_{xy}^{\text{BA}}(\mathbf{k}) f_{xy}^\dagger(\mathbf{k}) f_{xy}(\mathbf{k}) \right], \quad (18)$$

where  $E_{\text{d}}^{\text{BA}}(\mathbf{k}) = \sqrt{\epsilon_{\mathbf{k}}[\epsilon_{\mathbf{k}} + 2n(c_0 - Q|c_1|)]}$ ,  $E_z^{\text{BA}}(\mathbf{k}) = \sqrt{\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}} + q)}$  and  $E_{xy}^{\text{BA}}(\mathbf{k}) = \sqrt{(\epsilon_{\mathbf{k}} + 2n|c_1|)[\epsilon_{\mathbf{k}} + 2(1 - \tilde{q}^2)n|c_1|]}$  are interpreted to be the density, the gapless- and gapful-spin mode, respectively. In addition, the QZ perturbation is represented as

$$\mathcal{F}_{\text{Q}}^{\text{BA}} = \bar{\mathcal{F}}_{\text{Q}}^{\text{BA}} + \sum_{\mathbf{k} \neq 0} \left[ -\mathcal{A}_{\mathbf{k}}^{\text{BA}} f_z^\dagger(\mathbf{k}) f_z^\dagger(-\mathbf{k}) + \mathcal{B}_{\mathbf{k}}^{\text{BA}} f_{xy}^\dagger(\mathbf{k}) d(\mathbf{k}) + \mathcal{C}_{\mathbf{k}}^{\text{BA}} f_{xy}^\dagger(\mathbf{k}) f_{xy}^\dagger(-\mathbf{k}) + \mathcal{D}_{\mathbf{k}}^{\text{BA}} f_{xy}^\dagger(\mathbf{k}) d^\dagger(-\mathbf{k}) + \text{H.c.} \right], \quad (19)$$

where  $\bar{\mathcal{F}}_{\text{Q}}^{\text{BA}}$  commutes with the Hamiltonian (18), and the factors are  $\mathcal{A}_{\mathbf{k}}^{\text{BA}} = \sqrt{(\epsilon_{\mathbf{k}} + \tilde{q}n|c_1|)^2 - (E_z^{\text{BA}}(\mathbf{k}))^2}/2E_z^{\text{BA}}(\mathbf{k})$ ,  $\mathcal{B}_{\mathbf{k}}^{\text{BA}} = -\frac{\sin 2\theta}{4}[\sqrt{\alpha_{\mathbf{k}}\beta_{\mathbf{k}}} + \frac{1}{\sqrt{\alpha_{\mathbf{k}}\beta_{\mathbf{k}}}}]$ ,  $\mathcal{C}_{\mathbf{k}}^{\text{BA}} = -\frac{\tilde{q}}{4}[\alpha_{\mathbf{k}} - \frac{1}{\alpha_{\mathbf{k}}}]$ , and  $\mathcal{D}_{\mathbf{k}}^{\text{BA}} = \frac{\sin 2\theta}{4}[\sqrt{\alpha_{\mathbf{k}}\beta_{\mathbf{k}}} - \frac{1}{\sqrt{\alpha_{\mathbf{k}}\beta_{\mathbf{k}}}}]$  with  $\alpha_{\mathbf{k}} = E_{xy}^{\text{BA}}(\mathbf{k})/(\epsilon_{\mathbf{k}} + 2n|c_1|)$  and  $\beta_{\mathbf{k}} = E_{\text{d}}^{\text{BA}}(\mathbf{k})/\epsilon_{\mathbf{k}}$ . From Eq. (19), the EAR spectrum is given as  $R^{\text{BA}}(\omega) = (\delta q)^2 \mathcal{R}_{\text{c}}[r_z^{\text{BA}}(\omega) + \theta_{\text{H}}(|\tilde{\omega}| - \Delta_{\tilde{q}}^{\text{BA}})r_{\text{d}}^{\text{BA}}(\omega)]$  with

$2\Delta_{\tilde{q}}^{\text{BA}}r_{xy}^{\text{BA}}(\omega) + \theta_{\text{H}}(|\tilde{\omega}| - \Delta_{\tilde{q}}^{\text{BA}})r_{\text{d}}^{\text{BA}}(\omega)]$  with

$$r_z^{\text{BA}}(\omega) = \tilde{q}^2 \sqrt{\frac{\sqrt{\tilde{q}^2 + \tilde{\omega}^2} - \tilde{q}}{\tilde{q}^2 + \tilde{\omega}^2}}, \quad (20)$$

$$r_{xy}^{\text{BA}}(\omega) = \tilde{q}^6 \sqrt{\frac{\sqrt{\tilde{q}^4 + \tilde{\omega}^2} + \tilde{q}^2 - 2}{\tilde{q}^4 + \tilde{\omega}^2}}, \quad (21)$$

$$r_{\text{d}}^{\text{BA}}(\omega) = (2|\tilde{c}_1|)^{3/2} (\Delta_{\tilde{q}}^{\text{BA}})^2 |\tilde{\omega}| \times (|\tilde{\omega}| - \Delta_{\tilde{q}}^{\text{BA}})^2 \left[ \sqrt{\gamma(\omega)} - \frac{1}{\sqrt{\gamma(\omega)}} \right]^2, \quad (22)$$

where  $\gamma(\omega) = \frac{\Delta_{\tilde{q}}^{\text{BA}}}{(|\tilde{\omega}| - \Delta_{\tilde{q}}^{\text{BA}})[|\tilde{c}_1|^2(|\tilde{\omega}| - \Delta_{\tilde{q}}^{\text{BA}})^2 + |\tilde{c}_1|]}$ .  $\Delta_{\tilde{q}}^{\text{BA}} = \sqrt{1 - \tilde{q}^2}$  denotes the normalized gap of the spin mode  $E_{xy}^{\text{BA}}$ , and the normalized coupling  $\tilde{c}_1 = c_1/c_0$  has been introduced.

Equation (22) for the various  $\tilde{q}$ 's is shown in Fig. 2(c). The gapless spectral weight  $r_z^{\text{BA}}(\omega)$  describes a two-particle excitation of the gapless-spin mode  $E_z^{\text{BA}}(\mathbf{k})$ , and at the limit  $\tilde{q} \rightarrow 1$  it is identical to  $R^{\text{TP}}(\omega)$  at  $\tilde{q} \rightarrow 1$ . The weight  $r_{xy}^{\text{BA}}(\omega)$ , which is the two-particle excitation of the spin mode  $E_{xy}^{\text{BA}}(\mathbf{k})$ , gives the gapful EAR spectrum with the gap,  $2\Delta_{\tilde{q}}^{\text{BA}}$ , and quadratically vanishes at the limit  $\tilde{q} \rightarrow 0$ . The other spectrum weight  $r_{\text{d}}^{\text{BA}}(\omega)$  from the pair excitation of the quasiparticles of the gapless-phonon  $E_{\text{d}}^{\text{BA}}(\mathbf{k})$  and of the gapful-spin mode  $E_z^{\text{BA}}(\mathbf{k})$  provides a gapful spectrum with the gap  $\Delta_{\tilde{q}}^{\text{BA}}$ . The two-particle excitation of the spin and density mode is peculiar to the BA phase as seen in the form of the QZ perturbation (19). In addition, the spectrum weight  $r_{\text{d}}^{\text{BA}}(\omega)$  is of order  $\sqrt{c_0/c_1}$  while the others are independent of  $c_0$ . Thus the EAR for  $c_0 \gg |c_1|$  is dominated by  $r_{\text{d}}^{\text{BA}}(\omega)$  in the frequency regime where  $r_{\text{d}}^{\text{BA}}(\omega)$  is finite.

*Trap effect.*— Based on the above results in the homogeneous case, we discuss the EAR spectrum for trapped systems by local density approximation. Then, the spectrum is calculated by taking the average of the local spectrum with the weight of the local density  $n(\mathbf{r})$ . The local EAR is obtained by replacing the mean density  $n$  by the local one  $n(\mathbf{r})$ . Since the density  $n$  accompanies the interaction constant  $c_0$  and  $c_1$  in all the results, it turns out that the inhomogeneity modifies the strength of the interactions: The effective interaction around the trap center behaves stronger, and weaker as going away from the center. Thus, if the trap center is in the polar phase and in the FM, the whole system exhibits polar and FM state, respectively, as expected from Fig. 1. In addition, since the gap of the EAR spectrum is independent of the density and the interaction, the gapful feature of the EAR in the polar phases should remain even in the trapped case. On the other hand, when the trap center is in the BA phase, the outward regime would be polar state from the phase diagram 1. However, since the EAR in the BA phase is characterized by the gapless spectrum, the qualitative feature is expected to be protected. From the above discussion, the EAR spectrum can be concluded

to characterize the phases of spin-1 BECs regardless of the presence of trap potentials.

*Conclusion.*— We have discussed the response of the spinor BECs to modulation of the magnetic field by using linear response theory, and found the EAR spectrum to access the correlation of the QZ term. By calculating the EAR of the spin-1 BECs, we have demonstrated the spectrum as a function of the modulation frequency to show the explicit difference in each phase. In addition, we have pointed out that our results are robust against the

trap effect. We note that while we have considered only spin-1 BECs in this paper, application to spin-2 BECs is straightforward.

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